Operations on Algebraic Terms

In the discussions below, x, y, and z represent **variables**; their values can vary. For example, the distance driven varies with elapsed time, a loan payment varies with the interest rate, a number in a series varies with each iteration, etc. In addition, a, b, and c represent **coefficients** and **constants**. Coefficients are the numbers in front of the variables that tell how much of the variable we have, e.g. the 3 in $3x$. Constants are terms without variables; they don't change their values, e.g. 1, -5, etc.

What is an algebraic term?

An algebraic term can be thought of as having

- a sign (positive or negative, unless the term is zero, which is neither positive nor negative),
- a coefficient (since we're considering the sign separately, the coefficient is a non-negative number), and
- zero or more variables.

If no number is present in front of the variable(s), the coefficient is 1; for example, the coefficient of x is 1.

You can think of the sign and coefficient as being one unit, or you can think of them as separate units; I find that I consider the sign and coefficient as one unit when adding and subtracting algebraic terms and as separate units when doing other operations.

Algebraic term	Sign	Coefficient	Variables
x	$+$	1	1 variable: x
$3x$	$+$	3	1 variable: x
1	$+$	1	none
0	NA	0	none
$4x^2$	$+$	4	1 variable: x^2
$-xy^2z^3$	$-$	1	3 variables: x, y^2, and z^3
$\frac{-6}{x^2y}$	$-$	6	2 variables: x^{-2} and y^{-1}
$\sqrt{8x^3}$	$+$	$\sqrt{8}$	1 variable: $\sqrt{x^3}$

NA: Not Applicable

In general, operations on algebraic terms involve the same 3 steps:

1. Operate on the signs.
2. Operate on the coefficients.
3. Operate on the variable(s).

Exponent rules for algebraic terms

Addition and subtraction of like terms*	$ax^n \pm bx^n = (a \pm b)x^n$		No change in exponents
Multiplication	$ax^m \cdot bx^n = abx^{m+n}$		Addition of exponents
Division	$\frac{1}{x^n} = x^{-n}$	$\frac{ax^m}{bx^n} = \frac{a}{b}x^{m-n}$	Subtraction of exponents
Raise to nth power	$x^0 = 1$	$\left(ax^m\right)^n = a^n x^{mn}$	Multiplication of exponents
Take nth root	$\sqrt[n]{x} = x^{\frac{1}{n}}$	$\sqrt[n]{ax^m} = a^{\frac{1}{n}} x^{\frac{m}{n}}$	Division of exponents**

**See factoring on page 4.*

*** Fractions are division:* $\frac{numerator}{denominator} = numerator \div denominator$.

Before we get to examples of operations on algebraic terms, let's first examine what is meant by "like terms."

What are "like terms"?

Like terms are terms that have the same variables raised to the same powers (exponents); for example, two terms with x^2y (and only those variables) are like terms. When determining which terms are like terms, only the variables and their powers matter; the signs and coefficients don't matter. If an equation has like terms that are being added and subtracted, those terms can be simplified to one term by adding or subtracting their coefficients as their signs dictate; the variable portion of the result is the same as the variable portion of the terms. Unlike terms can *never* be added or subtracted together.

Like terms	Unlike terms
$10, 3, -8, 1$ (all are constants)	5 and x (5 is a constant and x is a variable)
$2x, 5x, -7x, 9002x$ (all have x raised to the 1st power)	x and x^2 (x is raised to different powers)
$x^2, -11x^2, -2x^2, \frac{1}{2}x^2$ (all have x raised to the 2nd power)	$3x$ and $3x^2$ (x is raised to different powers)
$4xyz^3, 7xyz^3, -xyz^3, 10xyz^3$ (each term has x raised to the 1st power, y raised to the 1st power, and z raised to the 3rd power)	xyz and xy^2z (at least one of the variables is raised to a different power)
$\frac{1}{x}, \frac{-8}{x}, \frac{2}{3x}, 5x^{-1}$ (all have x raised to the -1 power)	x^1 and x^{-1} (x is raised to different powers)

Expression	Are there any like terms?	Explanation
x^2+3x+1	No	x^2 is the only term that has x raised to the 2nd power, so it can't be added to or subtracted from any of the other terms. $3x$ is the only term with x raised to the 1st power, so it can't be added to or subtracted from any of the other terms. 1 is the only constant term, so it can't be added to or subtracted from any of the other terms.
$2x+3x-4x+7-2$	Yes	The terms $2x$, $3x$, and $-4x$ all have the variable x raised to the 1st power, so they can be combined into one term. These terms simplify to $2x+3x-4x=(2+3-4)x=x$. The terms 7 and -2 are both constants, so they can be combined into one term: $7-2=5$. The entire expression simplifies to $x+5$.
x^4-2x^3	No	x^4 is the only term that has x raised to the 4th power, so it can't be added to or subtracted from any of the other terms. $-2x^3$ is the only term that has x raised to the 3rd power, so it can't be added to or subtracted from any of the other terms.
$x-5x^2-8x+x^2$	Yes	The x and $-8x$ terms both have x raised to the 1st power, so they can be combined into one term: $x-8x=(1-8)x=-7x$. The $-5x^2$ and x^2 terms both have x raised to the 2nd power, so they can be combined into one term: $-5x^2+x^2=(-5+1)x^2=-4x^2$. The entire expression simplifies to $-4x^2-7x$.
x^2+2x^2	Yes	The terms x^2 and $2x^2$ both have x raised to the 2nd power, so they can be combined into one term: $x^2+2x^2=(1+2)x^2=3x^2$.

Examples of operations on algebraic terms

	Original Expression	(1) Operate on the signs[1]	(2) Operate on the coefficients	(3) Operate on the variable(s)	Final answer
Addition and Subtraction (no change in exponents)	$3x+5x$	$3+5=8$		x	$8x$
	$2x^2yz^5-6x^2yz^5$	$2-6=-4$		x^2yz^5	$-4x^2yz^5$
	$\frac{2}{3}x^3+\frac{3}{4}x^3-\frac{1}{2}x^3$	$\frac{2}{3}+\frac{3}{4}-\frac{1}{2}=\frac{8+9-6}{12}=\frac{11}{12}$		x^3	$\frac{11}{12}x^3$
Multiplication (addition of exponents)	$(3x^2)(4x^3y)$	all $+$, so $+$	$3\cdot4=12$	$(x^2)(x^3y)=x^{2+3}y=x^5y$	$12x^5y$
	$(2x)(-7xy)(xyz)$	odd $-$, so $-$	$2\cdot7\cdot1=14$	$(x)(xy)(xyz)=x^{1+1+1}y^{1+1}z=x^3y^2z$	$-14x^3y^2z$
	$(-x^5)(-23x)$	even $-$, so $+$	$1\cdot23=23$	$(x^5)(x)=x^{5+1}=x^6$	$23x^6$
Division (subtraction of exponents)	$\frac{-12x^3}{3x}$	odd $-$, so $-$	$\frac{\overset{4}{\cancel{12}}}{\underset{1}{\cancel{3}}}=4$	$\frac{x^3}{x}=x^{3-1}=x^2$ ***or*** $\frac{x^{\overset{2}{\cancel{3}}}}{\underset{1}{\cancel{x}}}=x^2$	$-4x^2$
	$y^5\div y^3$	all $+$, so $+$	$1\div1=1$	$y^5\div y^3=y^{5-3}=y^2$	y^2
	$\frac{42x^5y^4z}{7x^3y^5z}$	all $+$, so $+$	$\frac{\overset{6}{\cancel{42}}}{\underset{1}{\cancel{7}}}=6$	$\frac{x^5y^4z}{x^3y^5z}=x^{5-3}y^{4-5}z^{1-1}=x^2y^{-1}=\frac{x^2}{y}$ ***or*** $\frac{x^{\overset{2}{\cancel{5}}}\,\overset{1}{\cancel{y^4}}\,\overset{1}{\cancel{z}}}{\underset{1}{\cancel{x^3}}\,y^{\overset{1}{\cancel{5}}}\,\underset{1}{\cancel{z}}}=\frac{x^2}{y}$	$\frac{6x^2}{y}$
Raise to n^{th} power (multiplication of exponents)	$(5x^4)^2$	$(+)^{even}$, so $+$	$5^2=25$	$(x^4)^2=x^{4\cdot2}=x^8$	$25x^8$
	$(-2x^4y^3z)^3$	$(-)^{odd}$, so $-$	$2^3=8$	$(x^4y^3z)^3=x^{4\cdot3}y^{3\cdot3}z^{1\cdot3}=x^{12}y^9z^3$	$-8x^{12}y^9z^3$
	$(-4x)^2$	$(-)^{even}$, so $+$	$4^2=16$	$(x)^2=x^{1\cdot2}=x^2$	$16x^2$
Take n^{th} root (division of exponents)	$\sqrt{9x^2}$	$\sqrt[even]{+}$, so $+$	$\sqrt{9}=3$	$\sqrt{x^2}=x^{\frac{2}{2}}=x^1=x$	$3x$
	$\sqrt[3]{-8x^2y^5z}$	$\sqrt[odd]{-}$, so $-$	$\sqrt[3]{8}=2$	$\sqrt[3]{x^2y^5z}=x^{\frac{2}{3}}y^{\frac{5}{3}}z^{\frac{1}{3}}$ ***or*** $y\sqrt[3]{x^2y^2z}$	$-2y\sqrt[3]{x^2y^2z}$
	$\sqrt{\frac{16x^4}{y^{10}}}$	$\sqrt[even]{+}$, so $+$	$\sqrt{16}=4$	$\sqrt{\frac{x^4}{y^{10}}}=\sqrt{x^4y^{-10}}=x^{\frac{4}{2}}y^{\frac{-10}{2}}=x^2y^{-5}=\frac{x^2}{y^5}$	$\frac{4x^2}{y^5}$

[1] **Addition and subtraction:** When adding like signs (all positive or all negative signs), add the numbers together and keep the sign. When adding unlike signs, subtract the numbers and take the sign of the larger number.
Multiplication and division: Multiplication and division of all positive signs or an even number of negative signs results in a positive answer. Multiplication and division of an odd number of negative signs results in a negative answer.
Raised to a power: A positive number raised to any power and a negative number raised to an even power results in a positive product. A negative number raised to an odd power results in a negative product. This is, of course, a restatement of the multiplication rule above.
Taking a root: Any root of a positive number is positive. The odd root of a negative number is negative. The even root of a negative number is imaginary (uses $i=\sqrt{-1}$), which we won't discuss here.

Distributive Property

The **Distributive Property** is usually written in its most simple form as $a(b+c)=ab+ac$. It is important to understand that the Distributive Property is used <u>whenever terms that are added or subtracted are multiplied by "outside" factors</u>. For example, the Distributive Property must be used to expand the following expressions:

$$a(b+c-d+e-f+g)=ab+ac-ad+ae-af+ag$$

$$xy(w-z+5)=wxy-xyz+5xy$$

$$(x+y)^2=(x+y)(x+y)=x^2+xy+xy+y^2=x^2+2xy+y^2$$

$$(x+y)(w+x+y-z)=xw+x^2+xy-xz+wy+xy+y^2-yz=x^2+xw+2xy-xz+wy-yz+y^2$$

The Distributive Property also serves another very important purpose. It tells us how to factor:

distribute

$$a(b+c)=ab+ac$$

factor

Written in its factoring form, the Distributive Property is $ab+ac=a(b+c)$. When terms that are added or subtracted contain a common factor, that common factor can be **divided** out of each term and placed outside of parentheses (which, as usual, indicates multiplication of every term in the parentheses). The following are some examples:

$3x+9=3x+3\cdot 3=3(x+3)$ Another way of thinking about this is $3x+9=3\left(\frac{3x+9}{3}\right)=3\left(\frac{\cancel{3}^{1}x}{\cancel{3}_{1}}+\frac{\cancel{9}^{3}}{\cancel{3}_{1}}\right)=3(x+3)$.

$$7x^4y^3z-42x^3y^3z^2=7x^3xy^3z-7\cdot 6x^3y^3zz=7x^3y^3z(x-6z)$$

$$\sqrt{16-4x}=\sqrt{4\cdot 4-4x}=\sqrt{4(4-x)}=2\sqrt{4-x}$$

$$x^2+2x-(7x+14)=xx+2x-(7x+7\cdot 2)=x(x+2)-7(x+2)$$

Now we have a common factor of $x+2$, so we factor that out: $x(x+2)-7(x+2)=(x+2)(x-7)$

$$8x-3x+4x=(8-3+4)x=9x$$

Notice that this last example is simply the addition and subtraction of like algebraic terms; the variable portion of each term is factored out and the coefficients are then simplified.

The Distributive Property is used extensively in algebra in both its distributive and factoring forms. For example, the following simplification uses both distribution and factorization:

$$\frac{3x(x-1)+4(7x+2)-x(x+20)-20}{2x-3}=\frac{3x^2-3x+28x+8-x^2-20x-20}{2x-3}=\frac{2x^2+5x-12}{2x-3}=\frac{\cancel{(2x-3)}^{1}(x+4)}{\cancel{2x-3}_{1}}=x+4$$

Additional Notes

1. The Distributive Property is not the only rule you need to be able to use "forwards" and "backwards." Probably one of the most overlooked simplifications is not recognizing that the rule $(ab)^n = a^n b^n$ also works "in reverse": $a^n b^n = (ab)^n$. Some examples that benefit from using this rule "in reverse" are

$$(16x)^5(8x)^{-5} = \frac{(16x)^5}{(8x)^5} = \left(\frac{\cancel{16}^{\,2}\ \cancel{x}^{\,1}}{\cancel{8}_{\,1}\ \cancel{x}_{\,1}}\right)^5 = 2^5 = 32$$

$$(6x)^{\frac{1}{2}}(3x)^{-\frac{1}{2}} = \frac{(6x)^{\frac{1}{2}}}{(3x)^{\frac{1}{2}}} = \left(\frac{\cancel{6}^{\,2}\ \cancel{x}^{\,1}}{\cancel{3}_{\,1}\ \cancel{x}_{\,1}}\right)^{\frac{1}{2}} = \sqrt{2}$$

$$\sqrt[3]{9x}\cdot\sqrt[3]{3x^2} = \sqrt[3]{9x\cdot 3x^2} = \sqrt[3]{3^3x^3} = 3x$$

$$(6x)^{\frac{1}{2}}(3x)^{\frac{1}{2}} = (6x\cdot 3x)^{\frac{1}{2}} = \left(2\cdot 3^2\cdot x^2\right)^{\frac{1}{2}} = 3x\sqrt{2}$$

Spotting these types of simplifications becomes easier with practice. In general, you need to be able to apply any rule in its "forwards" form and in its "backwards" form.

2. Negative exponents are often difficult at first. For example, if given $\dfrac{a^{-n}x^m}{b^q y^{-r}}$ to simplify, you may write

$$\frac{a^{-n}x^m}{b^q y^{-r}} = \frac{\left(\frac{1}{a^n}\right)x^m}{b^q\left(\frac{1}{y^r}\right)} = \frac{\left(\frac{x^m}{a^n}\right)}{\left(\frac{b^q}{y^r}\right)} = \frac{x^m}{a^n}\div\frac{b^q}{y^r} = \frac{x^m}{a^n}\cdot\frac{y^r}{b^q} = \frac{x^m y^r}{a^n b^q}$$

, and you'd be correct. However, with practice, you'll write $\dfrac{a^{-n}x^m}{b^q y^{-r}} = \dfrac{x^m y^r}{a^n b^q}$ directly because you'll be comfortable with moving terms raised to negative powers to the opposite side of the fraction line. When in doubt though, always go the long route.

3. It is customary to write variables (a) in alphabetical order within terms and (b) in descending order of exponents. This helps to identify like terms and reduce errors. For example, if we write $(2x+3y)(4y-x) = 8xy - 2x^2 + 12y^2 - 3yx$, we may overlook the like terms of $8xy$ and $-3yx$ and not simplify our answer; the correct answer is $(2x+3y)(4y-x) = -2x^2 + 5xy + 12y^2$. Notice that all of the x terms are written first and in descending order of exponents, followed by the y term. Organization and consistency are greatly rewarded in math.

4. Check your work as you go. As you work a problem, undo your current step and make sure you get your previous step. If you don't, you've changed the problem—and that's bad!

5. If an equation looks confusing or you are uncertain how to proceed, always ask yourself **"What do the rules say?"**

6. If you are unsure of a rule, test it using small numbers, typically -1, 0, 1, 2, 3, or 5. For example, if given $\dfrac{x+6}{x+3}$ and you're confused on whether you can cancel the 6 and the 3, try it with $x=1$: Plugging into the original equation, you get $\dfrac{1+6}{1+3} = \dfrac{7}{4}$; if you cancel the 6 and the 3, you get $\dfrac{x+\cancel{6}^{\,2}}{x+\cancel{3}_{\,1}} = \dfrac{x+2}{x+1} = \dfrac{1+2}{1+1} = \dfrac{3}{2}$. Since $\dfrac{3}{2} \neq \dfrac{7}{4}$, you know you can't cancel the 6 and the 3. Of course, when this happens, it's helpful to review the correct rule as soon as you can.

Warnings

The following table shows examples of common errors:

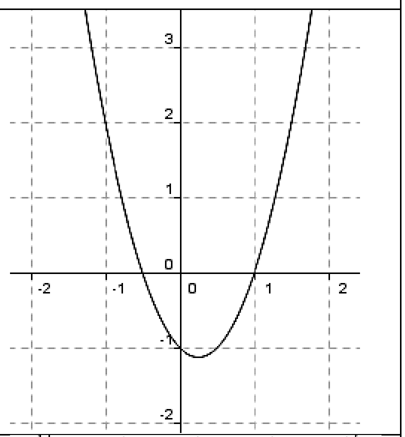
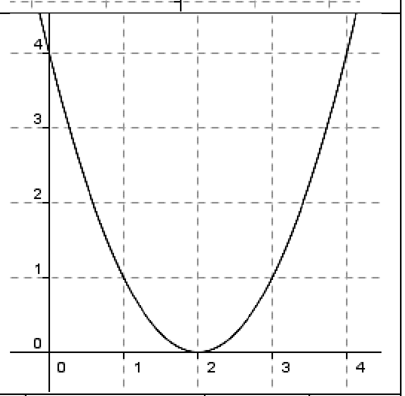
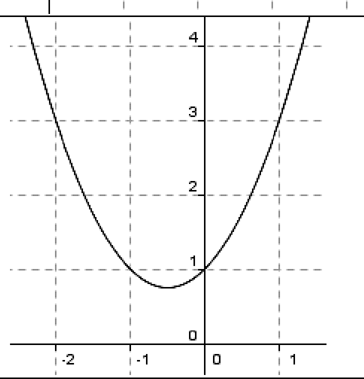
Expression	Pitfall and Correction
-3^2 and 4^{3^2}	**Common errors:** $-3^2 = 9$ and $4^{3^2} = \left(4^3\right)^2 = 4^6 = 4096$ Exponents only govern the terms they *directly* touch. Hence, in the expression -3^2, the negative sign is not governed by the exponent. The expression says to square 3 and then take its negative; the correct answer is $-3^2 = \boxed{-9}$. If we want the negative sign to be governed by the exponent, we must use parentheses: $(-3)^2 = 9$; now the exponent governs everything in the parentheses. In the second example, the exponent 2 only touches the exponent 3, so the correct answer is $4^{3^2} = 4^9 = \boxed{262144}$.
$(-3x)^2$ and $\left(\frac{4x}{5y}\right)^2$	**Common errors:** $(-3x)^2 = -3x^2$ and $\left(\frac{4x}{5y}\right)^2 = \frac{4^2x^2}{5y}$ An exponent must be applied to *all* terms inside the parentheses. The correct answer to the first example is $(-3x)^2 = (-3)^2\,x^2 = \boxed{9x^2}$; the correct answer to the second example is $\left(\frac{4x}{5y}\right)^2 = \frac{4^2x^2}{5^2y^2} = \boxed{\frac{16x^2}{25y^2}}$.
$x(x+4)$	**Common error:** $x(x+4) = x^2 + 4$ The Distributive Property must be used when terms that are added or subtracted are multiplied by "outside" factors; the correct answer is $x(x+4) = \boxed{x^2+4x}$.
$5x-4(x+1)$ and $7x+3-(5x-2)$	**Common errors:** $5x-4(x+1) = 5x-4x+4 = x+4$ and $7x+3-(5x-2) = 7x+3-5x-2 = 2x+1$ In the first example, the factor to be distributed within the parentheses is -4, not 4; the correct answer is $5x-4(x+1) = 5x-4x-4 = \boxed{x-4}$. In the second example, the factor to be distributed within the parentheses is -1; the correct answer is $7x+3-(5x-2) = 7x+3-5x+2 = \boxed{2x+5}$.
$(x+3)^2$	**Common error:** $(x+3)^2 = x^2+9$ The Distributive Property must be used when an exponent is applied to terms that are added or subtracted. Here, the term $x+3$ is multiplied by two "outside" factors, x and 3; the correct answer is $(x+3)^2 = (x+3)(x+3) = x(x+3)+3(x+3) = x^2+3x+3x+9 = \boxed{x^2+6x+9}$.
$\frac{x+6}{x+3}$ and $\sqrt{x^2+9}$	**Common errors:** $\frac{x+6}{x+3} = \frac{\overset{1}{\cancel{x}}+\overset{2}{\cancel{6}}}{\underset{1}{\cancel{x}}+\underset{1}{\cancel{3}}} = 2$ and $\sqrt{x^2+9} = x+3$ These cannot be simplified! Terms that are added or subtracted cannot be "broken up" and individually canceled or have their roots taken. Imagine a box around the addition or subtraction; you can only cancel or take the root of the box—you can never open the box and separately cancel or take the roots of its individual components. For example, you *can* do this $\frac{(3x+2)(x+4)}{3(x+4)} = \frac{\boxed{(3x+2)}\,\cancel{\boxed{(x+4)}}}{3\cancel{\boxed{(x+4)}}} = \frac{3x+2}{3}$ and this $\sqrt{\left(x^2+y^3+4\right)^2} = \sqrt{\boxed{\left(x^2+y^3+4\right)}^2} = x^2+y^3+4$.
$\sqrt[3]{9x}+\sqrt[3]{3x^2}$	**Common error:** $\sqrt[3]{9x}+\sqrt[3]{3x^2} = \sqrt[3]{27x^3} = 3x$ This cannot be simplified! If the 2 terms were multiplied together, we could put them under the same radical as we did earlier in the discussion of how $(ab)^n = a^nb^n$ also implies $a^nb^n = (ab)^n$. However, these 2 terms are *added* together and cannot be combined under a single radical. Since each of the 2 terms is already fully simplified, the entire expression cannot be further simplified.

Some important algebraic equations

Equation	Example
$(x+a)(x+b)=x^2+(a+b)x+ab$	$(x+2)(x+7)=x^2+9x+14$
$(x+a)^2=x^2+2ax+a^2$	$(x+5)^2=x^2+2(5)x+5^2=x^2+10x+25$
$(x-a)^2=x^2-2ax+a^2$	$(x-3)^2=x^2-2(3)x+3^2=x^2-6x+9$
$(x+a)(x-a)=x^2-a^2$ **Difference of Squares**	$(x+8)(x-8)=x^2-8^2=x^2-64$

Quadratic Formula:

The **roots** of the **quadratic equation** $ax^2+bx+c=0$ are given by the **quadratic formula** $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

If b^2-4ac ...	the root(s) are...	and the graph looks like this.
is positive	2 real numbers that identify the graph's 2 x-intercepts	
is zero	1 real number that identifies the graph's 1 x-intercept	
is negative	2 complex numbers (numbers with real and imaginary components) and there are no x-intercepts	

(The prefix *quad-* is normally used to indicate 4, as in *quad*rilateral, a 4-sided polygon. So why is the term *quadratic* used here? Quadratic is from the Latin *quadratum*, which means square; its use here references the fact that x^2 is the area of a square with sides of length x.)

Roots of $x^2+8x+15$ are

$$x=\frac{-8\pm\sqrt{8^2-4(1)(15)}}{2(1)}$$

$$x=\frac{-8\pm\sqrt{64-60}}{2}$$

$$x=\frac{-8\pm\sqrt{4}}{2}$$

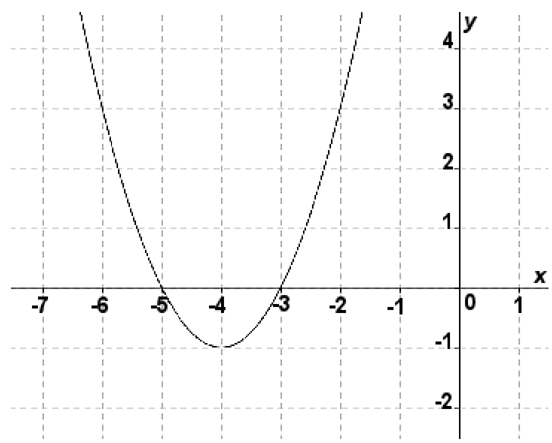
$$x=\frac{-8\pm 2}{2}$$

$$x=\frac{\cancel{2}^{1}(-4\pm 1)}{\cancel{2}_{1}}$$

$$x=-4\pm 1$$

$$x=-4+1 \text{ or } -4-1$$

$$x=-3 \text{ or } -5$$



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