Divisibility Rules and Their Explanations

Increase Your Number Sense

These divisibility rules apply to determining the divisibility of a positive integer (1, 2, 3, …) by another positive integer or 0 (although the divisibility rule for 0 says not to do it!)[1]. Explanations of the divisibility rules are included because understanding *why* a rule works helps you (1) to remember it and to use it correctly and (2) much more importantly, to deepen your knowledge of how numbers work. Some divisors have several divisibility rules; to limit the scope of this document, in most cases, only one divisibility rule is given for a divisor. A summary of the divisibility rules is included at the end.

The explanations for these divisibility rules are divided into the following categories:

Last Digits	**2^x 5^x 10^x**
Modular Arithmetic	**3 9 11**
Composite Number	**Composites**
Prime Number	**Primes**
Other	**0 1**

0 1 2 3 4 5 6 7 8 9 10 11 12 Composites Primes

Before we get to the divisibility rules, let's establish some groundwork required for the Last Digits and Modular Arithmetic explanations:

Foundations:

1. Any integer can be rewritten as the sum of its ones + its tens + its hundreds + …and so on (for simplicity, we will refer to an integer's ones, tens, hundreds, etc. as its "parts"). For example, the integer 3784 can be rewritten as the sum of its parts: 3000 + 700 + 80 + 4.

2. Dividing each of an integer's parts by a divisor and summing the results gives the same result as dividing the integer as a whole by the divisor. For example,

   a) 17 ÷ 4 = (10 ÷ 4) + (7 ÷ 4)

   $4\,^1/_4 = 2\,^2/_4 + 1\,^3/_4$

   $4\,^1/_4 = 3\,^5/_4$

   $4\,^1/_4 = 4\,^1/_4$

   b) 326 ÷ 2 = (300 ÷ 2) + (20 ÷ 2) + (6 ÷ 2)

   163 = 150 + 10 + 3

   163 = 163

[1] As always, if the dividend or divisor is negative, you do the division as though both were positive and figure out the sign of the quotient afterwards.

c) 549 ÷ 9 = (500 ÷ 9) + (40 ÷ 9) + (9 ÷ 9)

$61 = 55\,{}^{5}/_{9} + 4\,{}^{4}/_{9} + 1$

$61 = 60\,{}^{9}/_{9}$

$61 = 61$

3. An integer *d* is a divisor of an integer *N* if the remainder of *N* ÷ *d* is zero. When we are determining divisibility, we don't care how many times *d* goes into *N*; we only care if the remainder is zero. When dividing an integer by parts as shown in point 2 above, <u>divisibility means that the sum of the parts' remainders is divisible by the divisor</u>. There are 2 ways this can happen: Either each of the integer's parts is evenly divisible by the divisor as in Case b) above (the remainders sum to 0, which is divisible by the divisor 2) or each of the integer's parts is *not* evenly divisible by the divisor, but the sum of the remainders is as in Case c) above (5 + 4 + 0 = 9, which is divisible by the divisor 9).

4. If we know the remainder when 1 of a place value is divided by a divisor, we know something about the remainder for any amount of that place value divided by that divisor. For example, the remainder of 1000 ÷ 9 is 1; since 2000 ÷ 9 = (2 · 1000) ÷ 9 = 2 · (1000 ÷ 9), we can conclude that the remainder of 2000 ÷ 9 is 2 times the remainder of 1000 ÷ 9—and it is, 2. Sometimes when we apply this, we get a remainder that is larger than the divisor. For example, since 1000 ÷ 3 has a remainder of 1, using our approach to calculate the remainder of 5000 ÷ 3, we would calculate the remainder as 5 · 1 = 5, which is larger than 3. We can subtract off 3s from this odd-looking remainder (in essence, reduce it) until we get a number less than the divisor 3 and that would be the "true" remainder, in this case, 5 – 3 = 2. However, as far as divisibility rules are concerned, there's another way to think about this, as the divisibility rule for 3 explains…

5. If we add or subtract the divisor one or multiple times from an integer, we haven't changed if that integer is divisible by the divisor. For example, since 6 is divisible by 3, we can add or subtract one or more 3s from 6, and we're guaranteed to get a number that's divisible by 3, e.g. 6 + 3 = 9, 6 + (3 · 8) = 6 + 24 = 30, 6 – 3 = 3, and 6 – (3 · 10) = 6 – 30 = -24. On the other hand, since 5 is not divisible by 3, we can add or subtract one or more 3s from 5, and we're guaranteed to get a number that's not divisible by 3, e.g. 5 + 3 = 8, 5 + (3 · 8) = 5 + 24 = 29, 5 – 3 = 2, and 5 – (3 · 10) = 5 – 30 = -25. This may seem familiar because integer division is sometimes compared to repeated subtraction, i.e. counting how many times we can subtract the divisor from the dividend until a quantity less than the divisor remains. For example, 21 ÷ 4 = 21 – 4 – 4 – 4 – 4 – 4 with 1 left over, so 21 ÷ 4 = $5\,{}^{1}/_{4}$ because we can subtract the divisor 4 5 times until less than 4 remains (1); notice that no matter how many times we subtract 4, the result is never divisible by 4 (and, to analyze this further, the result always has a remainder of 1 if it's positive or a remainder of 1 – 4 = -3 if it's negative, i.e. if we subtract "too many" 4s). We will use this concept that additional or fewer instances of the divisor in a number don't affect divisibility several times in the Modular Arithmetic section.

<u>Notation Note</u>: To avoid confusion of the x multiplication symbol with the variable *x*, the dot (·) is used as the multiplication symbol in this document; for example, 3 times 2 is written as 3 · 2.

Last Digits Explanations

$2^x, 5^x, 10^x$ **For $x \geq 1$, an integer is divisible by 2^x, 5^x, or 10^x if its last x digits are.**

Specifically,
2^x: For $x \geq 1$, an integer is divisible by 2^x if its last x digits are divisible by 2^x.
5^x: For $x \geq 1$, an integer is divisible by 5^x if its last x digits are divisible by 5^x.
10^x: For $x \geq 1$, an integer is divisible by 10^x if its last x digits are 0.
Note that x is an integer, as implied by "x digits" since we cannot have a fraction of a digit.

Why? Let's analyze the base-10 (decimal) number system a little:

Table 1

Place value	**100,000**	**10,000**	**1000**	**100**	**10**	**1**
	10^5	10^4	10^3	10^2	10^1	10^0
	$(2 \cdot 5)^5 = 2^5 \cdot 5^5$	$(2 \cdot 5)^4 = 2^4 \cdot 5^4$	$(2 \cdot 5)^3 = 2^3 \cdot 5^3$	$(2 \cdot 5)^2 = 2^2 \cdot 5^2$	$(2 \cdot 5)^1 = 2^1 \cdot 5^1$	$(2 \cdot 5)^0 = 2^0 \cdot 5^0$
Exponent *x*	5	4	3	2	1	0

Important things to notice in Table 1:

(1) Because $10 = 2 \cdot 5$, in each place value column, 2, 5, and 10 are all raised to the same exponent; this is why the divisibility rules for 2^x, 5^x, and 10^x have the same form.

(2) Every place value ≥ the place value in column x is a multiple of 2^x, 5^x, and 10^x; for example, for the thousands column ($x = 3$), every place value ≥ 1000 is a multiple of 2^3, 5^3, and 10^3.

(3) A consequence of the exponent x starting at 0 is that the place value column in which 2^x, 5^x, and 10^x are located is always followed by x digits:

2^x
5^x
10^x $\underbrace{\qquad\qquad\qquad}_{x \text{ digits}}$

For example, the ten thousands column ($x = 4$) where 2^4, 5^4, and 10^4 are located has 4 place values that follow it (the thousands, hundreds, tens, and ones), and the tens column ($x = 1$) where 2^1, 5^1, and 10^1 are located has 1 place value that follows it (the ones).

Let's examine what happens when we divide a positive integer by 2^x, 5^x, or 10^x for $x \geq 1$; we'll do the division by dividing each of the integer's parts separately as described in the Foundations section. Because of point (2) above, the place values in columns x and higher will always have a remainder of 0 and won't contribute to the sum of the remainders—in short, they can be ignored because they are already known to be multiples of 2^x, 5^x, or 10^x. The x digits in the lower place values (those less than the x column's place value) are the only ones that *might not* be divisible by 2^x, 5^x, or 10^x, and hence the only digits we need to consider to determine divisibility.

For example, let's divide the integer 123,456 by $8 = 2^3$. Since all place values in columns $x \geq 3$ (the thousands place and higher) are multiples of 2^3, they have remainders of 0 when divided by 2^3; the only place values that are not *guaranteed* to be divisible by 2^3 are the hundreds, tens, and ones place values, so those are the ones we need to check. In short, we only need to ask if 456 is divisible by 8. We can do the division by dividing 456 as a whole by 8 or by parts as shown below:

456	=	400	+	50	+	6
remainder when ÷ 8		**0**		**2**		**6**
		these remainders are not guaranteed to be 0, so we need to check them				

Since the sum of the remainders is 2 + 6 = 8, which is clearly divisible by 8, 456 is divisible by 8, which means 123,456 is divisible by 8.

Let's look at 2^x, 5^x, and 10^x separately. Remember in these discussions that $x \geq 1$ for 2^x, 5^x, and 10^x.

2^x:

$2^1 = 2$: To determine divisibility by 2, we only need to ask if the last digit is divisible by 2.
$2^2 = 4$: To determine divisibility by 4, we only need to ask if the last 2 digits are divisible by 4.
$2^3 = 8$: To determine divisibility by 8, we only need to ask if the last 3 digits are divisible by 8.
$2^4 = 16$: To determine divisibility by 16, we only need to ask if the last 4 digits are divisible by 16.
etc.

For example, to determine if 2384 is divisible by 4, we only need to ask if 84 is divisible by 4; it is, so 2384 is. To determine if 2384 is divisible by 8, we only need to ask if 384 is divisible by 8; it is, so 2384 is.

This rule loses practical application pretty quickly—it's often difficult to look at a 5-digit integer and easily determine if it's divisible by 32 (2^5). Unless the original number is very, very large, it might be just as much trouble to carry out the original division as to use the divisibility rule.

Let's take a closer look at divisibility by 2:

2: An integer is divisible by 2 if it is even, i.e. its last digit is 0, 2, 4, 6, or 8.

Why? If you multiply any integer by 2, it will end in 0, 2, 4, 6, or 8—no other ones digits are possible. Conversely, every integer that ends in 0, 2, 4, 6, or 8 is a multiple of 2.

1 · 2 = **2**
2 · 2 = **4**
3 · 2 = **6**
4 · 2 = **8**
5 · 2 = **0**
6 · 2 = 1**2**
etc.

5^x:

$5^1 = 5$: To determine divisibility by 5, we only need to ask if the last digit is divisible by 5.
$5^2 = 25$: To determine divisibility by 25, we only need to ask if the last 2 digits are divisible by 25.
$5^3 = 125$: To determine divisibility by 125, we only need to ask if the last 3 digits are divisible by 125.
etc.

Notice that *powers* of 5 end in 5. (This is different than *multiples* of 5, which end in either 0 or 5 as noted below.)

For example, to determine if 2475 is divisible by 25, we only need to ask if 75 is divisible by 25; it is, so 2475 is. To determine if 2475 is divisible by 125, we only need to ask if 475 is divisible by 125; it isn't, so neither is 2475.

As with the rule for 2^x, this rule also loses practical application pretty quickly—it's often difficult to look at a 4-digit integer and easily determine if it's divisible by 625 (5^4). Unless the original number is very, very large, it might be just as much trouble to carry out the original division as to use the divisibility rule.

Let's take a closer look at divisibility by 5:

5: An integer is divisible by 5 if it ends in 0 or 5.

Why? If you multiply any integer by 5, it will end in either a 0 or a 5—no other ones digits are possible. Conversely, every integer that ends in 0 or 5 is a multiple of 5.

1 · 5 = **5**
2 · 5 = 1**0**
3 · 5 = 1**5**
4 · 5 = 2**0**
5 · 5 = 2**5**
etc.

<u>10^x:</u>

10^1 = 10: To determine divisibility by 10, we only need to ask if the last digit is 0.
10^2 = 100: To determine divisibility by 100, we only need to ask if the last 2 digits are 0.
10^3 = 1000: To determine divisibility by 1000, we only need to ask if the last 3 digits are 0.
10^4 = 10,000: To determine divisibility by 10,000, we only need to ask if the last 4 digits are 0.
etc.

When you multiply a number by the base of its number system, you in effect shift the number to the left and backfill the number with a zero. For example, if you multiply a number by the base of its number system 3 times, you in effect shift the number to the left 3 place values and append 3 zeros to the end of it. Since these divisibility rules apply to the base-10 (decimal) number system, multiplication of a number by 10 (the base) *r* times results in *r* zeros at the end of the number; for example, 45 · 10 · 10 · 10 = 45,000. Conversely, if a number ends in *r* zeros, we know it is divisible by 10 *r* times; for example, because 700 ends in 2 zeroes, we know that it is divisible by 10 2 times: 700 ÷ 10 = 70 and 70 ÷ 10 = 7.

Modular Arithmetic Explanations

Modular arithmetic is a great tool to develop and explain certain divisibility rules because the two disciplines share many characteristics. Both modular arithmetic and divisibility rules are concerned only with the remainders of division problems. Neither cares how many times an integer goes into another integer; they both only care about the remainder. For example, given that 34 ÷ 8 = 4 $^2/_8$, modular arithmetic is only concerned that the remainder is 2, while the divisibility rule for 8 is only concerned that the remainder is nonzero.

Specifically, the concepts we borrow from modular arithmetic for the explanations of these divisibility rules are

1. the approach of examining the pattern of remainders that results when incrementing powers of a number are divided by a certain divisor; for our purposes, we are interested in the pattern of remainders when place values, i.e. powers of 10, are divided by the number we want a divisibility rule for and
2. remainders that differ by a factor of the divisor are considered equivalent, which allows for unreduced remainders (see the divisibility rule for 3) and over-reduced (negative) remainders (see the divisibility rule for 11).

In a sense, modular arithmetic expands the ways we can represent and use remainders.

These explanations presume you have not encountered modular arithmetic before. I recommend that you read the explanations in the order given below, i.e. the explanation for division by 9 first, then the explanation for division by 3, and finally the explanation for division by 11.

Each explanation starts by dividing 111,111 by the given divisor; the integer 111,111 has 1 of each of the place values from 1 to 100,000 (enough for us to see any emerging pattern), so it is a particularly convenient number for us to use. As outlined in points 2 and 3 in the Foundations section, we'll sum the remainders of each part's division and check that sum for divisibility by the divisor to see if the original number is divisible by that divisor. Also, as described in point 4 of the Foundations section, we'll consider the remainder of each place value as simply being a multiple of its base place value's remainder.

9 **An integer is divisible by 9 if the sum of its digits is divisible by 9.**

Why?

111,111	=	100,000	+	10,000	+	1000	+	100	+	10	+	1
remainder when ÷ 9	=	**1** since 100,000 = 99,999 + 1 = (9 · 11,111) + 1	+	**1** since 10,000 = 9999 + 1 = (9 · 1111) + 1	+	**1** since 1000 = 999 + 1 = (9 · 111) + 1	+	**1** since 100 = 99 + 1 = (9 · 11) + 1	+	**1** since 10 = 9 + 1 = (9 · 1) + 1	+	**1** since 1 = 0 + 1 = (9 · 0) + 1

This pattern continues for all place values when divided by 9: There is always a remainder of 1. We sum the remainders and check the sum for divisibility by 9: Since the remainders sum to 6 in this case, which is not divisible by 9, 9 is not a divisor of 111,111.

But what about for other integers—for example, 2385? Using our approach, we get

2385	=	2000	+	300	+	80	+	5
remainder when ÷ 9	=	**2** since 2000 = 2 · 1000, so the remainder must be 2 times the remainder of 1000 ÷ 9	+	**3** since 300 = 3 · 100, so the remainder must be 3 times the remainder of 100 ÷ 9	+	**8** since 80 = 8 · 10, so the remainder must be 8 times the remainder of 10 ÷ 9	+	**5** since 5 = 5 · 1, so the remainder must be 5 times the remainder of 1 ÷ 9

We sum the remainders and check the sum for divisibility by 9: 2 + 3 + 8 + 5 = 18, which is divisible by 9, so 2385 is indeed divisible by 9.

In summary, the reason this divisibility rule for 9 works is because starting with the ones place value, every base place value has a remainder of 1 when divided by 9 and, as outlined in the Foundations section, because (1) we can break any integer into its parts, divide each part separately by the divisor, and sum the results, (2) consider only the sum of the remainders of those operations to determine divisibility, and (3) consider each of those remainders in terms of being a multiple of its base place value's remainder when divided by the divisor. Whew, that's a mouthful, but it just means that to determine divisibility we only need to look at the sum of the products of each digit multiplied by its base place value's remainder. Since each base place value's remainder when divided by 9 is 1, we multiply each digit by 1 and then sum the results, which is just the sum of the digits. If this sum of the remainders is divisible by the divisor, then so is the original number. To determine if an integer *N* is divisible by 9, we find that we can simply add up the digits of *N* and see if that sum is divisible by 9! Please make sure you understand this before reading further because the divisibility rules for 3 and 11 use the same reasoning.

Since this rule is only concerned with the sum of digits, it doesn't matter what order those digits are in. For example, we found that 2385 is divisible by 9, but we also found that *any* integer comprised of the digits 2, 3, 8, 5, and any number of zeroes is divisible by 9: 8352, 3285, 5238, 23,850, 80,500,030,002, etc.

The divisibility rule for 9 can be applied repeatedly. For example, suppose we are determining if 1,768,399,929,846,579 is divisible by 9. Using the divisibility rule, we get

$$1 + 7 + 6 + 8 + 3 + 9 + 9 + 9 + 2 + 9 + 8 + 4 + 6 + 5 + 7 + 9 =$$
$$1 + 2 + 3 + 4 + 5 + (2 \cdot 6) + (2 \cdot 7) + (2 \cdot 8) + (5 \cdot 9) =$$
$$15 + 12 + 14 + 16 + 45 =$$
$$102$$

Since we now want to know if 102 is divisible by 9, we can simply use the 9 divisibility rule again: 1 + 0 + 2 = 3. Since 3 is not divisible by 9, neither is 102, and therefore neither is 1,768,399,929,846,579.

To further show the power of the 9 divisibility rule, let's use our number sense and reconsider the addition we just did for 1,768,399,929,846,579, because we did way too much work! All we want to know is if the sum of the digits is divisible by 9—we don't really care what the sum is, just if 9 divides into it evenly. Therefore, a faster way to determine if 1,768,399,929,846,579 is divisible by 9 is to cast out any 9 digits and 9 sums since they will divide out evenly and contribute nothing to our knowledge of whether the sum is divisible by 9 (we could also cast out any products of 9, but that's often too much work—we only want the easy wins here). Let's look at the original number again and eliminate any 9 digits and 9 sums: **1**,**768**,**399**,**929**,8**4**6,**57**9. After eliminating the 9 digits and the 9 sums (1 + 8, 2 + 7, 3 + 6, and 4 + 5), we only need to sum the digits that are left: 8 + 6 + 7 = 21. We can reapply the divisibility rule for 9 (2 + 1 = 3, which is not divisible by 9) or just realize that 21 is not divisible by 9—either way, we did a whole lot less work than the first time! We worked smarter, not harder.

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3 An integer is divisible by 3 if the sum of its digits is divisible by 3.

Why?

111,111	=	100,000	+	10,000	+	1000	+	100	+	10	+	1
remainder when ÷ 3	=	**1** since 100,000 = 99,999 + 1 = (3 · 33,333) + 1	+	**1** since 10,000 = 9999 + 1 = (3 · 3333) + 1	+	**1** since 1000 = 999 + 1 = (3 · 333) + 1	+	**1** since 100 = 99 + 1 = (3 · 33) + 1	+	**1** since 10 = 9 + 1 = (3 · 3) + 1	+	**1** since 1 = 0 + 1 = (3 · 0) + 1

This pattern continues for all place values when divided by 3: There is always a remainder of 1. We sum the remainders and check the sum for divisibility by 3: Since the remainders sum to 6 in this case, which is divisible by 3, 3 is a divisor of 111,111.

But what about for other integers—for example, 2385? Using our approach, we get

2385	=	2000	+	300	+	80	+	5
remainder when ÷ 3	=	**2** since 2000 = 2 · 1000, so the remainder must be 2 times the remainder of 1000 ÷ 3	+	**3** since 300 = 3 · 100, so the remainder must be 3 times the remainder of 100 ÷ 3	+	**8** since 80 = 8 · 10, so the remainder must be 8 times the remainder of 10 ÷ 3	+	**5** since 5 = 5 · 1, so the remainder must be 5 times the remainder of 1 ÷ 3

Here is another way of thinking about remainders as alluded to in the Foundations section: We can consider the remainders above in red to be unreduced remainders. In much the same way we can use any convenient representation of one-half in an equation without changing the validity (0.5, 0.500, $^1/_2$, $^4/_8$, $^5/_{10}$, etc.), as far as determining divisibility is concerned, we can use unreduced remainders without changing the validity. These remainders are unreduced because they still have instances of the divisor in them; for example, the remainder 8 shown in red above is 2 (the reduced remainder) plus 2 instances of the divisor 3, i.e. 2 + 3 + 3.

The key to understanding this divisibility rule is to consider the remainder of each place value as simply being a multiple of its base place value's remainder. For example, we want to consider the remainder of 40 ÷ 3 in terms of (4 · 10) ÷ 3 = 4 · (10 ÷ 3), which means the remainder is 4 · 1 = 4, i.e. 4 times the remainder of 10 ÷ 3. When we use the associative property in this way to get a multiple of the base place value's remainder, we sometimes get an unreduced remainder, but as far as determining divisibility is concerned, that's fine since additional instances of the divisor don't affect divisibility.

Given that we can accept remainders ≥ 3 here, we now sum the remainders and check the sum for divisibility by 3:
2 + 3 + 8 + 5 = 18, which is divisible by 3, so 2385 is indeed divisible by 3. (Note that this works with reduced remainders too: 2 + (3 – 3) + (8 – 3 – 3) + (5 – 3) = 2 + 0 + 2 + 2 = 6, which is divisible by 3.)

This divisibility rule for 3 works for the same reason the divisibility rule for 9 works: Starting with the ones place value, every base place value has a remainder of 1 when divided by 3 and because of the same reasons outlined in the Foundations section. Again, it just means that to determine divisibility we only need to look at the sum of the products of each digit multiplied by its base place value's remainder. Since each base place value's remainder when divided by 3 is 1, we multiply each digit by 1 and then sum the results, which is just the sum of the digits. If this sum of the remainders is divisible by the divisor, then so is the original number. The only difference between the divisibility rule for 3 and the divisibility rule for 9 is that here we allow unreduced remainders to gain the benefit that each of the remainders for a given place value is equal to the digit in that place value. To determine if an integer *N* is divisible by 3, we find that we can simply add up the digits of *N* and see if that sum is divisible by 3!

Since this rule is only concerned with the sum of digits, it doesn't matter what order those digits are in. For example, we found that 2385 is divisible by 3, but we also found that *any* integer comprised of the digits 2, 3, 8, 5, and any number of zeroes is divisible by 3: 8352, 3285, 5238, 23,850, 80,500,030,002, etc.

The divisibility rule for 3 can be applied repeatedly. For example, suppose we are determining if 1,768,399,929,846,579 is divisible by 3. Using the divisibility rule, we get

1 + 7 + 6 + 8 + 3 + 9 + 9 + 9 + 2 + 9 + 8 + 4 + 6 + 5 + 7 + 9 =
1 + 2 + 3 + 4 + 5 + (2 · 6) + (2 · 7) + (2 · 8) + (5 · 9) =
15 + 12 + 14 + 16 + 45 =
102

Since we now want to know if 102 is divisible by 3, we can simply use the 3 divisibility rule again: 1 + 0 + 2 = 3. Since 3 is divisible by 3, so is 102, and therefore so is 1,768,399,929,846,579.

To further show the power of the 3 divisibility rule, let's use our number sense and reconsider the addition we just did for 1,768,399,929,846,579, because we did way too much work! All we want to know is if the sum of the digits is divisible by 3—we don't really care what the sum is, just if 3 divides into it evenly. Therefore, a faster way to determine if 1,768,399,929,846,579 is divisible by 3 is to cast out any 3 digits, sums of 3, and products of 3 since they will divide out evenly and contribute nothing to our knowledge of whether the sum is divisible by 3. Let's look at the original number again and eliminate any 3 digits, sums of 3, and products of 3: **1**,**768**,**399**,**929**,**846**,**579**. After eliminating the 3 digits, the sums of 3 (1 + 2), and the products of 3 (9 digits, 6 digits, 1 + 8, 2 + 7, 4 + 5, 8 + 7), we only need to sum the digits that are left: There aren't any digits left, so the number is divisible by 3. We did a whole lot less work than the first time! We worked smarter, not harder.

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11 **An integer is divisible by 11 if the sum of every other one of its digits subtracted from the sum of the remaining digits is divisible by 11.**

Why?

111,111	=	100,000	+	10,000	+	1000	+	100	+	10	+	1
remainder when ÷ 11	=	**10** since 100,000 = 99,990 + 10 = (11 · 9090) + 10	+	**1** since 10,000 = 9999 + 1 = (11 · 909) + 1	+	**10** since 1000 = 990 + 10 = (11 · 90) + 10	+	**1** since 100 = 99 + 1 = (11 · 9) + 1	+	**10** since 10 = 0 + 10 = (11 · 0) + 10	+	**1** since 1 = 0 + 1 = (11 · 0) + 1

This pattern continues for all place values when divided by 11: The remainders are always an alternating pattern of 1s and 10s. Let's think about those remainders of 10. When we say that 12 ÷ 11 has a remainder of 1, we mean that 12 has 1 extra beyond a whole multiple of 11; in other words, 12 has 1 too many to be a multiple of 11. So it seems natural to consider 10 as having 1 too few to be a multiple of 11. Since 1 too many is indicated by a positive remainder (+1), it also follows naturally that 1 too few is indicated by a negative remainder (-1). Further consideration shows that negative remainders are essentially over-reduced remainders. For example, analyzing 10 ÷ 11 again, if we keep reducing the remainder of 10 (which is less than the divisor, hence over-reducing), as we subtract off 11s, we get negative remainders: 10 – 11 = -1, -1 – 11 = -12, etc., and as far as determining divisibility is concerned, these are all perfectly fine (i.e., equivalent) remainders of 10 ÷ 11.

With this in mind, we can now reanalyze 111,111 ÷ 11, and where we have a remainder of 10, we'll substitute the equivalent remainder of 10 – 11 = -1:

111,111	=	100,000	+	10,000	+	1000	+	100	+	10	+	1
remainder when ÷ 11	=	**-1** since 100,000 = 100,001 – 1 = (11 · 9091) - 1	+	**1** since 10,000 = 9999 + 1 = (11 · 909) + 1	+	**-1** since 1000 = 1001 – 1 = (11 · 91) - 1	+	**1** since 100 = 99 + 1 = (11 · 9) + 1	+	**-1** since 10 = 11 – 1 = (11 · 1) - 1	+	**1** since 1 = 0 + 1 = (11 · 0) + 1

So now we have a wonderful pattern of alternating positive and negative 1s. The only way for 11 to be a divisor of 111,111 is if the sum of the remainders is divisible by 11. Since the remainders sum to 0 in this case, which is divisible by 11, 11 is a divisor of 111,111.

But what about for other integers—for example, 2385? Using our approach, we get

2385	=	2000	+	300	+	80	+	5
remainder when ÷ 11	=	**-2** since 2000 = 2 · 1000, so the remainder must be 2 times the remainder of 1000 ÷ 11	+	**3** since 300 = 3 · 100, so the remainder must be 3 times the remainder of 100 ÷ 11	+	**-8** since 80 = 8 · 10, so the remainder must be 8 times the remainder of 10 ÷ 11	+	**5** since 5 = 5 · 1, so the remainder must be 5 times the remainder of 1 ÷ 11

We now sum the remainders and check the sum for divisibility by 11: -2 + 3 + (-8) + 5 = -2, which is not divisible by 11, so 2385 is not divisible by 11.

In summary, the reason this divisibility rule for 11 works is because starting with the ones place value, when base place values are divided by 11, the remainders alternate between 1 and -1 (since we allow over-reduced remainders) and because of the same reasons outlined in the Foundations section. Once again, it just means that to determine divisibility we only need to look at the sum of the products of each digit multiplied by its base place value's remainder. Since each base place value's remainder when divided by 11 is ±1, we in effect sum every other digit and subtract the sum of the remaining digits. If the sum of the remainders is divisible by the divisor, then so is the original number. To determine if an integer *N* is divisible by 11, we find that we can simply add up the digits of *N* with alternating positive and negative signs and see if that sum is divisible by 11!

Unlike the divisibility rules for 3 and 9, the order of the digits does matter for the divisibility rule by 11 because of the alternating positive and negative signs. For example, using the digits 6, 3, 8, and 5, we find that 6385 is not divisible by 11 (since -6 + 3 -8 + 5 = -6), but 3685 is divisible by 11 (since -3 + 6 – 8 + 5 = 0).

The divisibility rule for 11 can be applied repeatedly, but because the procedure subtracts about as often as it adds, it's nearly always unnecessary. For example, suppose we are determining if 1,768,399,929,846,579 is divisible by 11. Using the divisibility rule, we get -1 + 7 - 6 + 8 - 3 + 9 - 9 + 9 - 2 + 9 - 8 + 4 - 6 + 5 - 7 + 9 = 18. We know that 18 is not divisible by 11, but just to prove the point, we can use the 11 divisibility rule again: -1 + 8 = 7. Since 7 is not divisible by 11, neither is 18, and therefore neither is 1,768,399,929,846,579. Even with a number maximized to get a large sum, e.g. 90,909,090,909,090,909, the procedure yields 9 · 9 = 81, which is clearly not a multiple of 11.

If all we want to know is divisibility by 11, it doesn't matter if we alternate positive and negative signs starting with the ones place (which is technically the correct starting point) or starting with the highest place value. You may find it easier to start with the highest place value, especially when mentally calculating this rule. Earlier we found that 2385 is not divisible by 11 because -2 + 3 – 8 + 5 = -2; however, we could have alternated positive and negative signs beginning with the thousands place value (in essence, multiply the equation -2 + 3 – 8 + 5 by -1) and gotten 2 – 3 + 8 – 5 = 2. Divisibility by 11 means that the remainders will eventually sum to zero, and 0 · -1 is still zero.

The caveat with alternating positive and negative signs starting with the highest place value is if you really want to know what the remainder of the division is, not just if 11 divides evenly into the number. If you have been particularly observant, you may have noticed that all of these remainders we've been getting for these modular arithmetic explanations are, in fact, the remainders of the original numbers divided by the divisor in question (or at least some form of the remainder). For example, just above, we found that 2385 is not divisible by 11 because we calculated a remainder of -2. Remember that in modular arithmetic, a negative remainder means too few to be a multiple of the divisor; hence, a remainder of -2 when dividing by 11 means the remainder is equivalently 2 less than 11, or 11 – 2 = 9. If we divide 2385 by 11, guess what remainder we get? Yes, 9.

If we disregard the mathematically rigorous starting point (the ones place) for alternating positive and negative signs for this rule (which, again, is fine if all we want to know is if 11 divides the number evenly), we will get the negative of the true remainder if the original number has an even number of digits. We just calculated an example of this: Instead of getting a remainder of -2 for 2385 ÷ 11, we got a remainder of +2 when we alternated positive and negative signs starting with the highest place value. If we had really wanted to know the remainder of 2385 ÷ 11, we would've errored and said the remainder was 2, instead of 9. In practice, we're almost always interested in just determining divisibility, not in finding the actual remainder. This is just a heads up if you ever are interested in the remainder's actual value.

Composite Number Explanations

A composite number is a number that has factors other than 1 and itself.

Composite numbers **An integer is divisible by a composite number divisor if it is divisible by the highest power of each of the composite number's prime factors.**

Why?
If the divisor is a composite number, we need to find its prime factorization and test for divisibility by the highest power of each of its prime factors—this ensures that *all* of an integer's prime factors are tested. For example, when dividing an integer by 20, it's a mistake to think, "20 is 2 · 10, so I can just check if the integer is divisible by 2 and by 10." 20's prime factorization is $2^2 \cdot 5 = 2 \cdot 2 \cdot 5$. Because 10 is not a prime number (10 = 2 · 5), checking for divisibility by 10 is the same as testing for divisibility by 2 *and* by 5 in one step; in essence, we've tested for 2 of 20's prime factors: $2 \cdot \not{2} \cdot \not{5}$. If we then, in a separate step, test the integer for divisibility by 2, we're just repeating the test for divisibility for 2 that we did when we tested for divisibility by 10—the 2nd 2 never gets tested. This leads to errors: 10, 30, 50, 70, etc. are all divisible by 2 and by 10, but they aren't divisible by 20. Testing the highest power of each prime factor is the only way to ensure the entire composite number divisor has been tested.

To illustrate, the divisibility rules for 6, 12, 72, and 343 are given below:

6: An integer is divisible by 6 if it is divisible by both 2 and 3, i.e. if it is an even number that is divisible by 3.
Why? Since 6's prime factorization is 2 · 3, any integer divisible by both 2 and 3 is also divisible by 6.

For example, 2385 is divisible by 3, but not by 2 (since it is not an even number); therefore, 2385 is not divisible by 6. 2394 is an even number that is divisible by 3, so it is divisible by 6.

12: An integer is divisible by 12 if it is divisible by both 3 and 4.
Why? Since 12's prime factorization is $2^2 \cdot 3$, any integer divisible by both 3 and 4 is also divisible by 12.

For example, 2556 is divisible by 3 and by 4; therefore, 2556 is also divisible by 12.

72: An integer is divisible by 72 if it is divisible by both 8 and 9.
Why? Since 72's prime factorization is $2^3 \cdot 3^2$, any integer divisible by both 8 and 9 is also divisible by 72.

For example, 2556 is divisible by 9, but not by 8; therefore, 2556 is not divisible by 72.

343: An integer is divisible by 343 if it is divisible by...343 (not much help, is it?).
Why? Since $343 = 7^3$, this divisibility rule resolves to a truism: An integer is divisible by 343 if it is divisible by 343. Composite divisors that factor to only a prime raised to a power don't benefit from this divisibility rule.

Prime Number Explanations

A prime number is a number whose only factors are 1 and itself.

Prime numbers except 2 and 5

"Subtract *that* or add the complement."

******* **For ease of reading, throughout the Prime Number Explanations section, "prime number" and "*p*" refer to any prime <u>except</u> 2 and 5.** *******

The general procedure to check for divisibility of integer *N* by prime number *p* ($p \neq 2, 5$, as noted above) is

1. Delete any ending zeros of *N* to form a new *N*.
2. Multiply the last digit of *N* by a multiplier *m* and add or subtract it from the rest of *N*. (How to determine *m* and whether to add or subtract is explained below.)
3. Stop if you recognize that the result is or is not divisible by the divisor *p*.
4. Repeat Steps 1 through 3 until you stop in Step 3.

Examples

Calculation shown in detail	What the calculation looks like in practice
For 2385 ÷ 7, *N* = 2385, *p* = 7, and *m* = -2: 2385 238 + (5 · -2) = 238 - 10 = 228 22 + (8 · -2) = 22 - 16 = 6 6 is not divisible by 7, so neither is 2385.	2 3 8 5 - 1 0 2 2 8 - 1 6 6
For 850,369 ÷ 13, *N* = 850,369, *p* = 13, and *m* = 4: 850,369 85,036 + (9 · 4) = 85,036 + 36 = 85,072 8507 + (2 · 4) = 8507 + 8 = 8515 851 + (5 · 4) = 851 + 20 = 871 87 + (1 · 4) = 87 + 4 = 91 9 + (1 · 4) = 9 + 4 = 13 13 is divisible by 13, so 850,369 is.	8 5 0 3 6 9 + 3 6 8 5 0 7 2 + 8 8 5 1 5 + 2 0 8 7 1 + 4 9 1 + 4 1 3
For 374,890 ÷ 17, *N* = 374,890, *p* = 17, and *m* = -5: 374,890 37,489 3748 + (9 · -5) = 3748 – 45 = 3703 370 + (3 · -5) = 370 – 15 = 355 35 + (5 · -5) = 35 – 25 = 10 10 is not divisible by 17, so neither is 37,489.	3 7 4 8 9 0 - 4 5 3 7 0 3 - 1 5 3 5 5 - 2 5 1 0

In some cases, the divisibility rule for prime numbers may not be any faster than just doing the original division—you have to decide if using this rule is worth the effort. Also, note that unlike the modular arithmetic divisibility rules, this procedure does not result in the actual remainder of the original division problem.

Each prime number *p* actually has 2 multipliers, one negative (m_1) and one positive (m_2). Since m_1 is negative, you can think of the operation as subtraction instead of addition; the rule "subtract *that* or add the complement" does. The "subtract *that*" portion of the rule refers to the product of m_1 times *N*'s ones digit (with the subtraction sign already built in) and the "add the complement" portion refers to the product of m_2 times *N*'s ones digit. As the rule states, we only need to use one of those. Naturally, we want to use the easier of the 2 multipliers.

How to determine the multipliers m_1 and m_2

1. Determine the first multiple of the prime divisor *p* that ends in a 1.
2. Delete the 1 off that multiple to give a new number *k*.
3. $m_1 = -k$ and $m_2 = p - k$

Here are the calculations for the primes under 50 (excluding 2 and 5 since this rule doesn't apply to them); I've highlighted the multipliers I think are easier to use:

Prime number divisor *p*	1st multiple that ends in 1	*k*	m_1 (negative multiplier)	m_2 (positive multiplier)
3	3 · 7 = 21	2	**-2**	3 – 2 = 1
7	7 · 3 = 21	2	**-2**	7 – 2 = 5
11	11 · 1 = 11	1	**-1**	11 – 1 = 10
13	13 · 7 = 91	9	-9	13 – 9 = **4**
17	17 · 3 = 51	5	**-5**	17 – 5 = 12
19	19 · 9 = 171	17	-17	19 – 17 = **2**
23	23 · 7 = 161	16	-16	23 – 16 = **7**
29	29 · 9 = 261	26	-26	29 – 26 = **3**
31	31 · 1 = 31	3	**-3**	31 – 3 = 28
37	37 · 3 = 111	11	-11	31 – 11 = 20
41	41 · 1 = 41	4	**-4**	41 – 4 =37
43	43 · 7 = 301	30	-30	43 – 30 = 13
47	47 · 3 = 141	14	-14	43 – 14 = 29

(Note that here we give divisibility rules for 3 and 11 in addition to those discussed in the Modular Arithmetic section.)

In the 1st example of this section, I used $m_1 = -2$ for determining divisibility by 7; to me, that was the easier multiplier to use. I could've used the positive multiplier $m_2 = 5$ and gotten the same result:

For 2385 ÷ 7, *N* = 2385, *p* = 7, and *m* = 5:

2385
238 + (5 · 5) = 238 + 25 = 263
26 + (3 · 5) = 26 + 15 = 41
41 is not divisible by 7, so neither is 2385.

To summarize, this divisibility rule is phrased as "subtract *that* or add the complement" because the 2 ways to test for divisibility involve finding the tens and greater portion of the 1st multiple of *p* to end in the digit 1 and subtracting *that* times the last digit of *N* <u>or</u> adding the *p*'s complement of *that* times the last digit of *N* until you get a result that clearly is or is not divisible by *p*. Since I know "*that*" refers to the multiplier equal to *k*, the slant rhyme of "subtr**act** *th**at***" helps me to remember which multiplier is negative.

Why?

To get an idea of why this works for prime numbers (except 2 and 5), let's examine why the divisibility rule for 7 works. We begin by assuming a 3-digit integer *abc* (any number of digits works) is divisible by 7; *abc* must then be a multiple of 7:

Table 2

General Procedure	Example using 861 (= 7 x 123)	Notes
$abc = 7M$	$861 = 7M$	(1) Assume *abc* is a multiple of 7
$100a + 10b + c = 7M$	$100(8) + 10(6) + 1 = 7M$	(2) Break *abc* into parts
$100a + 10b + c - c = 7M - c$	$100(8) + 10(6) + 1 - 1 = 7M - 1$	(3) Subtract original number's ones digit *c* from both sides
$100a + 10b = 7M - c$	$100(8) + 10(6) = 7M - 1$	(4) The left-hand side is now *ab*0
$\frac{100a + 10b}{10} = \frac{7M - c}{10}$	$\frac{100(8) + 10(6)}{10} = \frac{7M - 1}{10}$	(5) Divide both sides by 10 to shift *ab*0 one place to the right
$10a + b = \frac{7M - c}{10}$	$10(8) + 6 = \frac{7M - 1}{10}$	(6) Simplify the left-hand side
$ab = \frac{7M - c}{10}$	$86 = \frac{7M - 1}{10}$	(7) Further simplify the left-hand side to *ab*
$ab - 2c = \frac{7M - c}{10} - 2c$	$86 - 2(1) = \frac{7M - 1}{10} - 2(1)$	(8) Subtract some quantity of *c* (the original number's ones digit) from both sides
$ab - 2c = \frac{7M - c}{10} - \frac{20c}{10}$	$86 - 2(1) = \frac{7M - 1}{10} - \frac{20(1)}{10}$	(9) Get a common denominator
$ab - 2c = \frac{7M - 21c}{10}$	$86 - 2 = \frac{7M - 21}{10}$	(10) Simplify the right-hand side; notice that the quantity of *c* that we subtracted from both sides in Step (8) has ensured that the quantity of *c* on the right-hand side of this step (21*c*) is divisible by 7
$ab - 2c = 7 \cdot \left(\frac{M - 3c}{10} \right)$ We have shown that subtracting 2 times *c* (the last digit of *abc*) from *ab* (the rest of the number) is a multiple of 7 <u>if the original number *abc* is a multiple of 7</u>.	(11) $86 - 2 = 7 \cdot \left(\frac{M - 3}{10} \right)$ Because 861 is a multiple of 7, there exists an integer *M* (123) that makes this equation true. On the right-hand side, $7 \cdot \left(\frac{123 - 3}{10} \right) = 7 \cdot \left(\frac{120}{10} \right) = 7 \cdot 12 = 84$ which equals the left-hand side of $86 - 2$. (12) In abbreviated form using only the left-hand side (i.e. using the rule): $86 - 2 = 84$ $8 - 2(4) = 8 - 8 = 0$ 0 is divisible by 7, therefore 84 is divisible by 7, which means that 861 is divisible by 7.	(11) Factor 7 out of the right-hand side; notice that we can only do this if Step (1) was valid; if our assumption in Step (1) is wrong, the right-hand side will not be divisible by 7 because there was never a valid integer *M* (12) Repeat this whole procedure again starting with *ab* – 2*c* until you recognize the left-hand side as divisible or not divisible by 7. ***This divisibility rule does not tell us what M is; the rule only tells us that IF the original number is divisible by the prime divisor, the left-hand side of this step is also divisible by the prime divisor. The value of this rule is that <u>we only need to check the left-hand side</u>.***

To emphasize the most important points noted above, in Step (11) we are able to factor out the prime divisor we're interested in (here, 7) because

- in Step (1) we assume the original number *abc* is a multiple of that divisor and
- on the right-hand side of the equation in Step (9), ignoring the denominators, we in effect add a factor of 10 *c* to the 1 *c* that is already there to get a quantity of *c* that is a multiple of the divisor (here, 20*c* + 1*c* = ***21c***).

The factor of 10 mentioned in the 2nd bullet is one of the 2 multipliers (m_1 or m_2) specific to each prime divisor *p*. For *p* = 7, m_1 = -2 as shown above; this multiplier is negative so that it adds to the "- *c*" term already on the right-hand side. The multiplier is always a "factor of 10" because in Step (5) we shift the *ab* digits to the right one place value which introduces a 10 in the denominator on the right-hand side, and in Step (9) we get a common denominator which results in the multiplier being multiplied by 10.

As stated above, the first multiplier m_1 is calculated to *add* to the 1 *c* on the right-hand side to result in a total quantity of *c* that is a multiple of *p*. The total quantity of *c* we end up with is $(10m_1 + 1)c$; hence, the total quantity of *c* will always end in the digit 1. Since we want that total quantity of *c* to be divisible by *p*, we need the first multiple of *p* that ends in the digit 1. We truncate the 1 off that multiple of *p*, and what remains is *k*, the factor of 10 *c* we need to subtract from both sides (see Step (8) above). $m_1 = -k$ because the multiplier needs to be negative so that it adds to the "- *c*" term already on the right-hand side.

The second multiplier m_2 is calculated to *subtract* the 1 *c* on the right-hand side to result in a total quantity of *c* that is a multiple of *p*. The total quantity of *c* we end up with is $(10m_2 - 1)c$; hence, the total quantity of *c* will always end in the digit 9. Since we want that total quantity of *c* to be divisible by *p*, we need to add 1 to the first multiple of *p* that ends in the digit 9 to offset the 1 *c* we're going to subtract. The result of adding 1 to a number that ends in 9 is we get a number that ends in 0, i.e. a factor of 10; we truncate the 0 off since m_2 is calculated to be a factor of 10, and what remains is … $p - k$! We can verify this: Since m_1 is derived from the 1st factor of *p* that ends in a 1 and m_2 is derived from the 1st factor of *p* that ends in a 9, the sum of these 2 factors ends in a 0 and is the 1st factor of 10 *p*; specifically, it's 10*p*. We already know the term that ends in 1: 10*k* + 1; if we subtract that from 10*p*, we get the term that ends in 9:

$$
\begin{aligned}
&Let\ f_1 = 1^{st}\ factor\ of\ p\ that\ ends\ in\ 1 \\
&\quad f_9 = 1^{st}\ factor\ of\ p\ that\ ends\ in\ 9 \\
&f_1 + f_9 = 10p \\
&f_9 = 10p - f_1 \\
&f_9 = 10p - (10k + 1) \\
&f_9 = 10p - 10k - 1 \\
&f_9 = 10(p - k) - 1
\end{aligned}
$$

Hence, to get the 1st multiple of *p* that ends in a 9, we need to subtract 1 from the factor of 10 that's equal to $(p - k)$. The upshot of all this is once we know *k*, we don't need to do the work of finding the first multiple of *p* that ends in a 9 to work our way to m_2; we can simply calculate m_2 directly as $m_2 = p - k$. m_2 needs to be positive so that it subtracts the "- *c*" term already on the right-hand side.

Examples of these calculations are shown below:

Prime *p*	**3**	**7**	**11**	**13**	**17**	**19**	**53**	**61**	**97**
k	2	2	1	9	5	17	37	6	29
p* - *k	1	5	10	4	12	2	16	55	68
f_1 **= 10*k* + 1** = 1st multiple of *p* that ends in 1	21	21	11	91	51	171	371	61	291
f_9 **= 10(*p* – *k*) – 1** = 1st multiple of *p* that ends in 9	9	49	99	39	119	19	159	549	679
10*p* = $f_1 + f_9$ = (10*k* + 1) + [10(*p* – *k*) – 1]	30	70	110	130	170	190	530	610	970

Notice that we'll always be able to find *both* a multiple of *p* that ends in 1 *and* a multiple of *p* that ends in 9 because, except for 2 and 5 (and this is why the rule doesn't apply to them), all prime numbers end in the digits 1, 3, 7, or 9—and those 4 digits multiply to products that end in 1 and in 9. For example, primes that end in 3 get multiplied by 7 to get their 1st product that ends in 1 and by 3 to get their 1st product that ends in 9.

We could've used m_2 in Table 2; as Step (13) below shows, we add (*p* – *k*)*c* to both sides (specifically, (7 – 2)*c* = 5*c*):

$$abc = 7M$$

$$100a + 10b + c = 7M$$

$$100a + 10b = 7M - c$$

$$10a + b = \frac{7M - c}{10}$$

$$10a + b + 5c = \frac{7M - c}{10} + 5c \qquad (13)$$

$$10a + b + 5c = \frac{7M - c}{10} + \frac{50c}{10} \qquad (14)$$

$$ab + 5c = \frac{7M + 49c}{10}$$

$$ab + 5c = 7 \cdot \left(\frac{M + 7c}{10}\right) \qquad (15)$$

This shows that adding 5 times *c* (the last digit of *abc*) to *ab* (the rest of the number) is a multiple of 7 if the original number *abc* is.

Notice that in Step (14), on the right-hand side of the equation, we in effect subtract the 1 *c* that is already there from a factor of 10 *c* to get a total quantity of *c* that is a multiple of *p* (specifically, 50*c* – 1*c* = 49*c*) and that this is what allows us to factor out a 7 in Step (15) if the original number is a multiple of 7.

Let's take a closer look at this rule and the prime number 3. m_1 is calculated in the usual way: The first multiple of 3 that ends in a 1 is 3 · 7 = 21. We truncate the 1, and therefore *k* = 2 and $m_1 = -k = -2$. If we use the shortcut method to calculate m_2 as $m_2 = p - k$, we get $m_2 = 3 - 2 = 1$. However, this isn't 1 more than the first multiple of 3 to end in a 9. If we calculate m_2 using that method, we get $m_2 = (3 \cdot 3) + 1 = 10$. Since m_2 is added to the tens and greater portion of the number we're testing, using $m_2 = 10$ means adding additional instances of 3 to check for divisibility by 3; it's as if we wanted to check if 6 is divisible by 3 by adding 9 to 6 and then checking if 15 is divisible by 3—adding a multiple of the divisor just makes the number larger and doesn't change that number's divisibility by the divisor. We can subtract m_2's additional instances of 3 without altering its ability to determine divisibility. We then get $m_2 = 10 - (3 \cdot 3) = 1$—the same value for m_2 we get using the shortcut method of $m_2 = p - k$.

Let's look at some examples comparing the Prime Number rule for 3 with m_2 = 1 and the Modular Arithmetic rule for 3:

Example using 18936		**Example using 34892**		**Example using *abcde***	
Prime Number rule for 3 using m_2 = 1	**Modular Arithmetic rule for 3**	**Prime Number rule for 3 using m_2 = 1**	**Modular Arithmetic rule for 3**	**Prime Number rule for 3 using m_2 = 1***	**Modular Arithmetic rule for 3**
1 8 9 3 6 + 6 1 8 9 9 + 9 1 9 8 + 8 2 7 + 7 9	1 + 8 + 9 + 3 + 6 = 27 2 + 7 = 9	3 4 8 9 2 + 2 3 4 9 1 + 1 3 5 0 + 5 8	3 + 4 + 8 + 9 + 2 = 26 2 + 6 = 8	*a b c d e* + *e* *a b c f* + *f* *a b g* + *g* *a h* + *h* *i* where $f = d + e$ $g = c + f$ $h = b + g$ $i = a + h$ Substituting back up the line gives $i = a + (b + g)$ $i = a + b + (c + f)$ $i = a + b + c + d + e$	$a + b + c + d + e$

* With liberties such as *f* – *i* possibly having 2 digits noted.

Although intermediate sums differ, the Prime Number rule for 3 using m_2 = 1 ultimately results in the same value as the Modular Arithmetic rule for 3 (repeated as necessary and without first casting out 3s and multiples of 3). For both rules, the question of divisibility hinges on the divisibility of that final value by 3. In effect, the Prime Number rule for 3 using m_2 = 1 and the Modular Arithmetic rule for 3 (without casting out) are the same rule! (For speed though, the fastest method of determining divisibility by 3 is the Modular Arithmetic rule with casting out.)

To summarize, the Prime Number rule works because whenever we use the procedure shown in Table 2 to test a positive integer's divisibility by a prime divisor *p* (except 2 and 5), we get the term $\frac{pM - d_1}{10} + \frac{10md_1}{10}$ where $m = m_1$ or m_2 and d_1 = the integer's ones digit, just like we do on the right-hand side in Steps (9) and (14) above. The value of *m* ensures the total quantity of d_1 on the right-hand side will be a multiple of *p*. If our assumption that the original number is a multiple of *p* is valid, the *pM* term will be valid, and we'll be able to factor out *p* in the last step. If our assumption that the original number is a multiple of *p* is invalid, it was invalid to set the equation equal to *pM* to start, and we won't be able to factor out *p* in the last step. In essence, *m* determines the quantity of d_1 we need to add to both sides of the equation to isolate the validity of the *pM* term. As long as we guarantee a multiple of pd_1 on the right-hand side, the validity of the whole equation falls to the validity of setting it equal to *pM* in the first place, i.e. "Is the integer a multiple of *p*?"—which is exactly what we set out to test. The divisibility rule for prime numbers is just a simplification of this whole process; in effect, it says we only need to check the left-hand side of the final result of Table 2's General Procedure, generalized for any given prime divisor (except 2 and 5) and its associated multiplier *m*.

Other Explanations

0 **By definition, division by 0 is undefined—a mathematical no-no.**

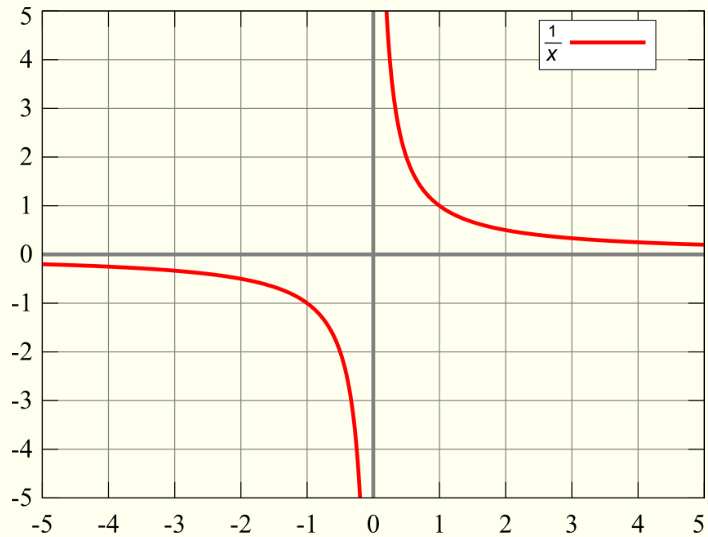
Why?

Division by 0 is undefined because it leads to contradictions and ill-defined conclusions:

a) Division and multiplication are reciprocal operations; in other words, if $c \div b = a$, then $a \cdot b = c$. If we allow $5 \div 0$ to equal some number n, then $n \cdot 0$ would = 5—but it doesn't ($n \cdot 0 = 0$), so we have a contradiction.

b) We don't really have a good way of thinking about what division by 0 means. For example, we understand what 1 ÷ 4 is—we might picture a circle divided into 4 equal parts, run a quarter mile lap, or notice that it's a quarter to 8; but what does 1 ÷ 0 mean? What does it mean to divide something into zero parts, to have one zeroth of something?

c) Advanced: As shown on the right, the graph of $^1/_x$ (or equivalently, $1 \div x$) has a discontinuity at zero. It goes to positive infinity when 0 is approached from the right and to negative infinity when 0 is approached from the left.



1 **All integers are divisible by 1.**

Why?

For any integer N, $N \cdot 1 = N$; therefore, every N is divisible by 1 ($N \div 1 = N$). In other words, 1 is a factor of every integer N, and therefore 1 is a divisor of every integer N.

Summary of Divisibility Rules	
0	By definition, division by 0 is undefined.
1	All integers are divisible by 1.
2	An integer is divisible by 2 if it is even, i.e. it ends in 0, 2, 4, 6, or 8.
3	An integer is divisible by 3 if the sum of its digits is divisible by 3.
4	An integer is divisible by 4 if the integer formed by its tens and ones digits is divisible by 4.
5	An integer is divisible by 5 if it ends in 0 or 5.
6	An integer is divisible by 6 if it is an even number that is divisible by 3.
7	An integer is divisible by 7 if the result of multiplying the last digit by -2 and adding it to the rest of the number is divisible by 7.
8	An integer is divisible by 8 if the integer formed by its hundreds, tens, and ones digits is divisible by 8.
9	An integer is divisible by 9 if the sum of its digits is divisible by 9.
10	An integer is divisible by 10 if it ends in at least 1 zero.
11	An integer is divisible by 11 if the sum of every other one of its digits subtracted from the sum of the remaining digits is divisible by 11.
12	An integer is divisible by 12 if it is divisible by both 3 and 4.
Composite	An integer is divisible by a composite number divisor if it is divisible by the highest power of each of the composite number's prime factors.
Prime except 2 and 5	"Subtract *that* or add the complement."
2^x	For $x \geq 1$, an integer is divisible by 2^x if its last x digits are divisible by 2^x.
5^x	For $x \geq 1$, an integer is divisible by 5^x if its last x digits are divisible by 5^x.
10^x	For $x \geq 1$, an integer is divisible by 10^x if it ends in at least x zeroes.